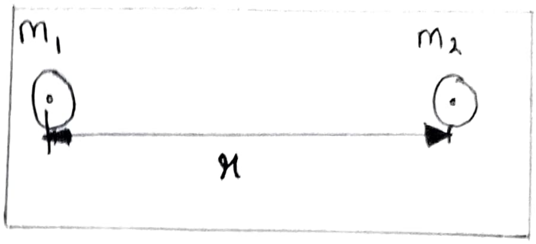
GRAVITATION

NEWTON'S LAW OF GRAVITATION → According to this law every particle of matter in the universe attracts every other particle with a force that is directly proportional to the product of masses and inversely proportional to the square of distance b/w them.

$F \propto m_1 m_2$



$F \propto \frac{1}{r^2}$

$F \propto \frac{m_1 m_2}{r^2}$

$F = \frac{G m_1 m_2}{r^2}$ Where G = Universal Gravitational Constant

$G = 6.67 \times 10^{-11}\ Nm^2\ kg^{-2}$

Important properties Of Gravitational force.

1/ It is conservative force which means work done by this force depends upon initial and final position of the body.

2/ It does not depend upon the nature of medium present b/w the two particle.

3/ It is long range force.

4/ It is Central force

5/ It is valid for point mass.

6/ It is always attractive

7/ It does not depend upon charge of the two bodies.

GRAVITY → In newton's law of gravitation, gravitation is the force of attraction b/w any two bodies.

But if one of the bodies is earth the gravitational force is called gravity.

UNIVERSAL LAW OF GRAVITION IN VECTOR FORM

$\vec{F}_{12}$ = Force on particle ① due to ②

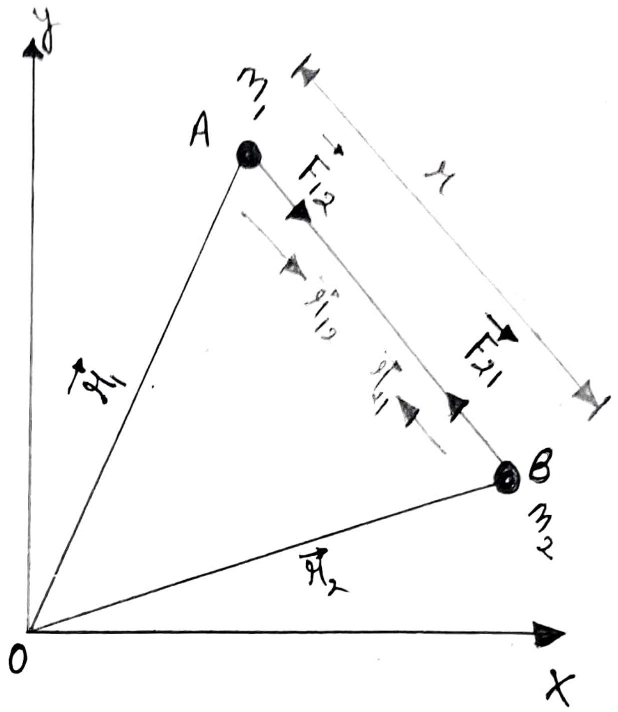
$\vec{F}_{21}$ = Force on particle ② due to ①

r = distance b/w two particle

$\vec{r}_{12}$ = Displacement vector from particle ① to ②

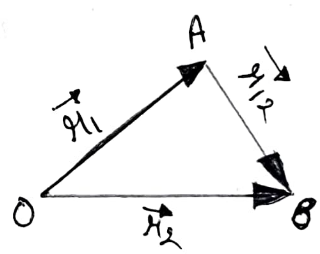
$\vec{r}_{21}$ = Displacement vector from ② to ①

$\vec{r}_1$ and $\vec{r}_2$ = Position vector of particle ① and ② respectively.



$$\vec{F}_{12} = \frac{G m_1 m_2}{r^2} \hat{r}_{12} \quad \text{— ①}$$

In $\triangle$ OAB



using triangle law of vector addition

$$\vec{r}_1 + \vec{r}_{12} = \vec{r}_2$$

$$\vec{r}_{12} = \vec{r}_2 - \vec{r}_1$$

$$\hat{r}_{12} = \frac{\vec{r}_{12}}{|\vec{r}_{12}|} = \frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|}$$

$$\Rightarrow \hat{r}_{12} = \frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|} \quad \text{— ②}$$

$$|\vec{r}_{12}| = r = |\vec{r}_2 - \vec{r}_1| \quad \text{— ③}$$

Put the value of equation ② and ③ in equation ①

$$\vec{F}_{12} = \frac{G m_1 m_2}{|\vec{r}_2 - \vec{r}_1|^2} \frac{(\vec{r}_2 - \vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|}$$

$$\vec{F}_{12} = \frac{G m_1 m_2 (\vec{r}_2 - \vec{r}_1)}{|\vec{r}_1 - \vec{r}_2|^3}$$

Similiarly $\vec{F}_{21} = \dfrac{G m_1 m_2 (\vec{r}_1 - \vec{r}_2)}{|\vec{r}_1 - \vec{r}_2|^3}$

Since $(\vec{r}_2 - \vec{r}_1) = -(\vec{r}_1 - \vec{r}_2)$

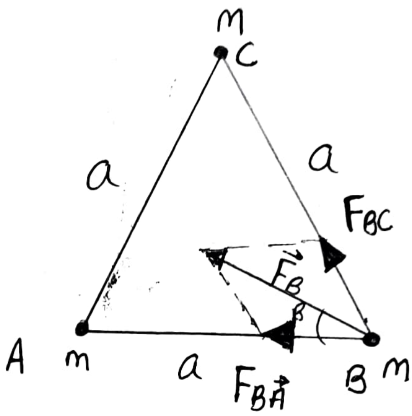
so $\vec{F}_{12} = -\vec{F}_{21}$

☆☆ GRAVITATION follows Newton third law of motion

NUMERICAL ON UNIVERSAL LAW OF GRAVITATION

Q→1 Three point Masses 'm' are placed at the vertices of an equilateral triangle of side m. Calculate net gravitational force on any of the three particle.

Sol The force on particle B to due to particle A and C will be



$$F_{BC} = F_{BA} = \frac{Gmm}{a^2} = \frac{Gm^2}{a^2} = F$$

$$\vec{F_B} = \vec{F_{BA}} + \vec{F_{BC}}$$

$$F_B = \sqrt{F_{BA}^2 + F_{BC}^2 + 2F_{BA} \times F_{BC}\cos 60^\circ}$$

$$F_B = \sqrt{F^2 + F^2 + 2FF\cos 60}$$

$$F_B = \sqrt{2F^2 + 2F^2\cos 60^\circ}$$

$$F_B = \sqrt{2F^2(1+\cos 60^\circ)}$$

$$F_B = \sqrt{2F^2\, 2\cos^2 30^\circ}$$

$$F_B = 2F\cos 30^\circ$$

$$F_B = 2F\frac{(\sqrt{3})}{2}$$

$$F_B = \sqrt{3}F$$

$$F_B = \frac{\sqrt{3}\,Gm^2}{a^2}$$

Magnitude of force

$$1+\cos 2\theta = 2\cos^2\theta$$
$$1+\cos 60^\circ = 2\cos^2 30^\circ$$

Direction of force

$$\tan\beta = \frac{F_{BC}\sin 60^\circ}{F_{BC}\cos 60^\circ + F_{AB}}$$

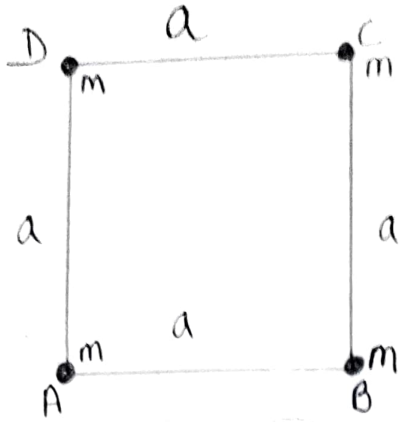
$$\tan\beta = \frac{F \times \frac{\sqrt{3}}{2}}{\frac{F}{2} + F}$$

$$\tan\beta = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}+1} = \frac{\frac{\sqrt{3}}{2}}{\frac{3}{2}}$$

$$\tan\beta = \frac{1}{\sqrt{3}}$$

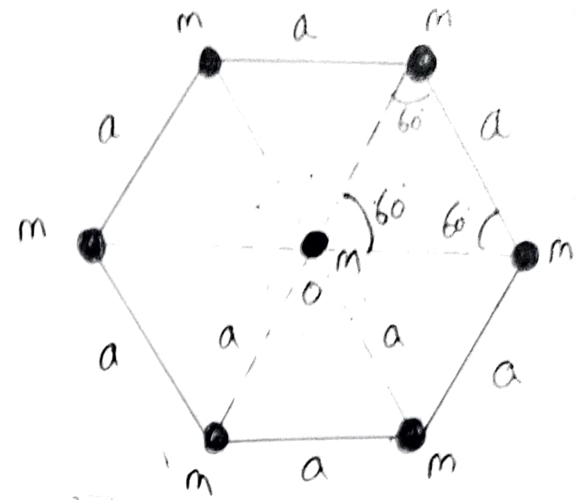
$$\beta = 30^\circ$$

Q→ Find net force on any one of the particle in following case

I 

Ans → $\left(\sqrt{2}+\frac{1}{2}\right)\frac{Gm^2}{a^2}$

(II) Calculate force on particle O

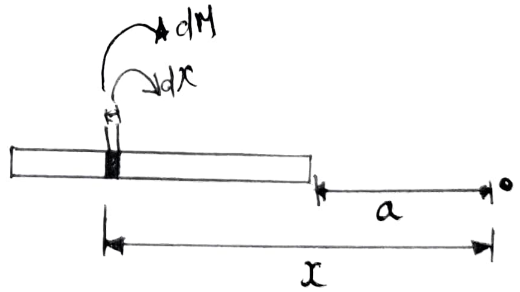


Ans:- $F = 0$

Net force on a particle placed at centre of a regular polygon will always be zero if all the masses kept at its vertices is same

Q→ A mass m is at distance a from one end of a uniform rod of length l and mass M. Find the gravitational force on mass 'm' due to rod.

Sol Let us consider a small element of thickness dx of mass dM at a distance x from the particle having mass m.



Force on m due to small element dx

$L \longrightarrow M$

$1 \longrightarrow \frac{M}{L}$

$dx \rightarrow dM \longrightarrow \frac{M}{L}dx$

$$dF = \frac{Gm(dM)}{x^2}$$

$$dF = \frac{G\left(\frac{M}{L}dx\right)m}{x^2}$$

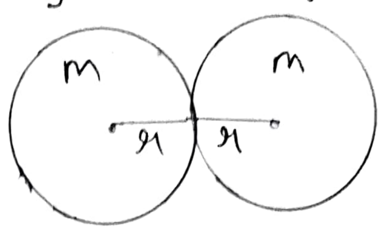
$$F = \frac{GMm}{L}\int_a^{a+L}\frac{1}{x^2}dx \Rightarrow F = \frac{GMm}{a(a+L)}$$

After integration

Q→ Sphere of same material and same radius r are touching each other. Show that gravitational force b/w them is directly proportional to r^4.

Sol

$$F = \frac{G\,mm}{(2r)^2}$$



$$F = \frac{GM^2}{4r^2}$$

$$m = \frac{4}{3}\pi r^3 \rho$$

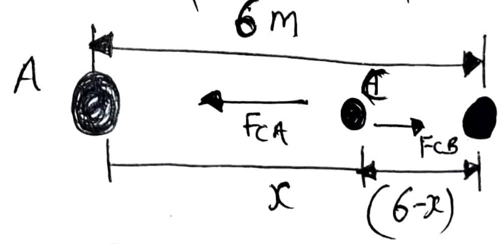
$$F = \frac{G}{4\cancel{r}^2}\left(\frac{4}{3}\pi r^3 \rho\right)^2$$

$$F = \frac{16\pi^2\rho^2 r^6}{9 \cancel{\times} 4 r^2} \Rightarrow F = \frac{4}{9}\pi^2\rho^2 r^4$$

$$F \propto r^4$$

Q→ Two particle of mass $2m$ and m are separated by a distance of $6m$. Find the point at which a particle of mass m is placed so that net force is zero on that particle.

Sol Since F_{net} on C is zero so



$$|\vec{F}_{CA}| = |\vec{F}_{CB}|$$

$$\frac{G(2m)m}{x^2} = \frac{G\,m \times m}{(6-x)^2}$$

$$2(6-x)^2 = x^2$$

$$\sqrt{2}(6-x) = \pm x$$

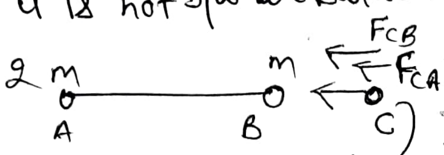
$$x = \frac{6\sqrt{2}}{1+\sqrt{2}}$$

$$x = 3.525\,m$$

$$x = \frac{6\sqrt{2}}{\sqrt{2}-1}$$

$$x = 20.48$$

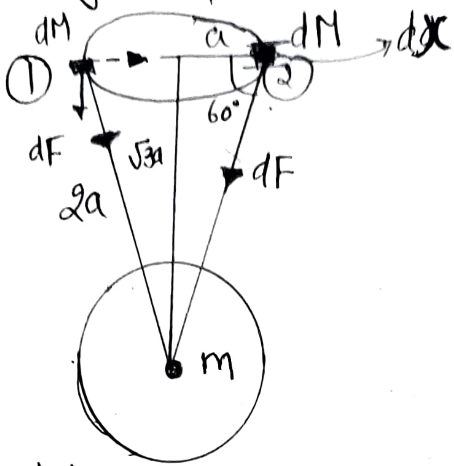
This answer can not be possible because it is not b/w two particle

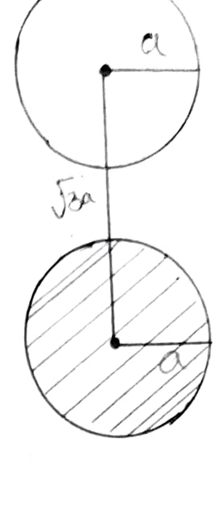


Both forces are in same direction so the will never make zero

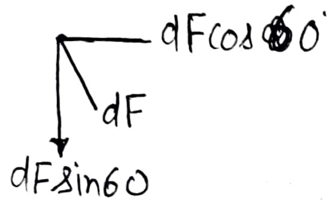
Q→ A uniform Ring of mass M is lying at a distance $\sqrt{3}a$ from the centre of sphere of mass m just over the sphere (where a is the radius of sphere as well as ring) Find the magnitude of gravitational force b/w them.

Sol

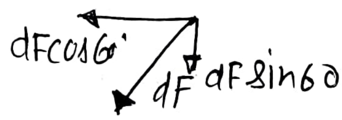




at particle (1) dM



at particle (2) dM



So the $dF\cos 60°$ component of two diametrically opp point will cancel out each other so only sin component will act as a force on ring.

$$F_{net} = \int_0^{2\pi a} dF\sin 60° = \frac{\sqrt{3}}{2}\int_0^{2\pi a} \frac{G\,dM\,m}{(2a)^2}$$

$$dM = \frac{M}{2\pi a}dx$$

$$= \frac{\sqrt{3}}{2}\frac{Gm}{4a^2}\int_0^{2\pi a} dM$$

$$= \frac{\sqrt{3}\,GMm}{2\times 4\times 2\pi a^3}\int_0^{2\pi a} dx$$

$$= \frac{\sqrt{3}GMm}{16\pi a^3}\Big[x\Big]_0^{2\pi a}$$

$$= +\frac{\sqrt{3}GMm}{16\pi a^3}\left[2\pi a - 0\right]$$

$$F_{net} = \frac{\sqrt{3}GMm}{8a^2}$$

ACCELERATION DUE TO GRAVITY [g]

When a body is dropped from certain height above the ground it begins to fall towards the earth under gravity.

The acceleration produced in the body due to gravity is known as acceleration due to gravity.

Suppose M is mass of earth and R is radius of earth then the force of attraction acting on a body of mass m close to the surface of earth

$$F = \frac{GMm}{R^2}$$

According to Newton's second law of motion

$$g = \frac{F}{m} = \frac{GM}{R^2}$$

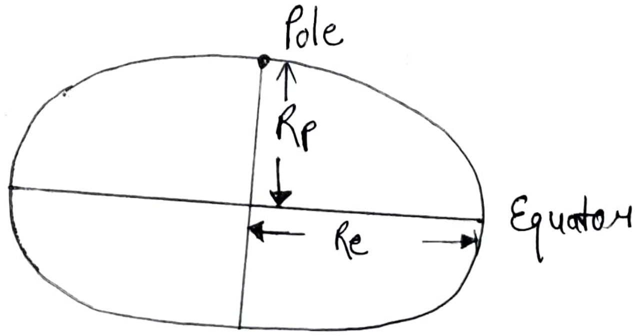
$$\boxed{g = \frac{GM}{R^2}}$$

VARIATION IN ACCELERATION DUE TO GRAVITY

① Shape Of EARTH

$$g = \frac{GM}{R^2}$$

$$g \propto \frac{1}{R^2}$$

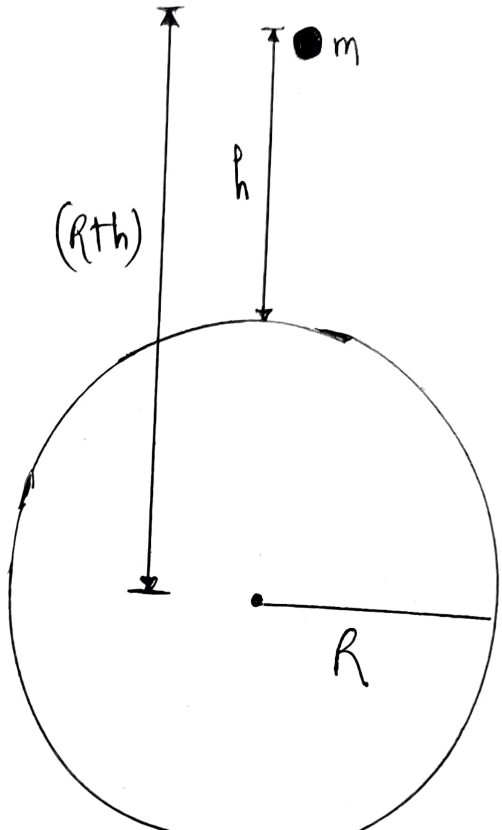


$$R_p < R_E$$

$$\boxed{g_P > g_E}$$

So value of g at pole is greater than value of g at equator

II Height above the surface of earth:



Value of acceleration due to gravity at earth surface

$$g = \frac{GM}{R^2} \quad \text{— (1)}$$

Value of acceleration due to gravity at height h

$$g' = \frac{GM}{(R+h)^2} \quad \text{— (2)}$$

Divide equation (2) with (1)

$$\frac{g'}{g} = \frac{\frac{GM}{(R+h)^2}}{\frac{GM}{R^2}}$$

$$g' = g\frac{R^2}{(R+h)^2}$$

$$g' = \frac{gR^2}{R^2\left(1+\frac{h}{R}\right)^2}$$

$$g' = \frac{g}{\left(1+\frac{h}{R}\right)^2} \Longrightarrow \boxed{g' = g\left(1+\frac{h}{R}\right)^{-2}}$$

This formula will be used if $h = 1200$ Km, $h = 3000$ Km

OR

$h = \frac{R}{2}$

$h = 2R$

if $h \lll R$

$$\boxed{g' = g\left(1-\frac{2h}{R}\right)}$$

{ using bionomial expansion

$(1+x)^m = (1+mx)$

$(1+x)^{-m} = (1-mx)$

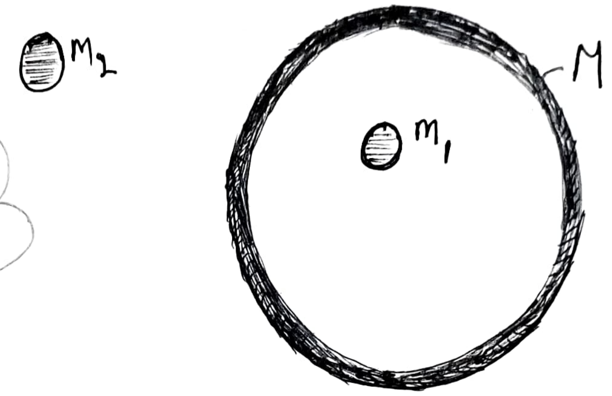
This formula will be used if $h = 20$ km

$h = 30$ km OR $h = 100$ km

Shell theorem ⇒ According to this theorem the gravitation force on a body placed inside the shell due to the shell will be zero.
[Gauss' theorem]

But the body inside the shell will not be shielded from the gravitational force of bodies situated outside the shell.

Force on m_1 due to M will be zero but it will not be zero due to M_2



III DEPTH BELOW THE SURFACE OF EARTH ⇒

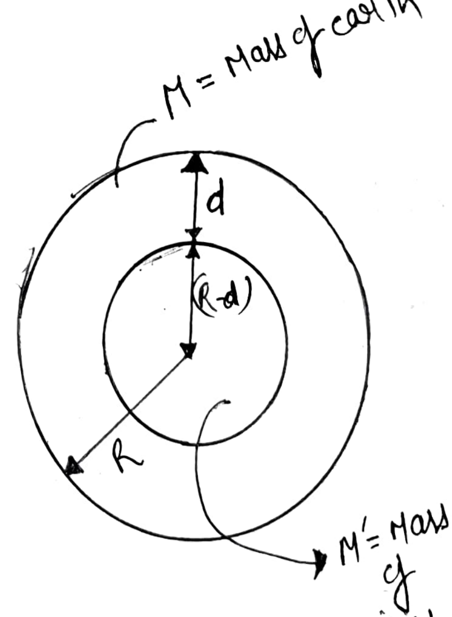
Acceleration due to gravity at the surface of Earth

$$g = \frac{GM}{R^2}$$

$$M = \frac{4}{3}\pi R^3 \rho$$

$$g = \frac{\frac{4}{3} G\pi R^3 \rho}{R^2}$$

$$\boxed{g = \frac{4}{3} G\pi R\rho} \quad (1)$$



Acceleration due to gravity at depth d

$$g' = \frac{GM'}{(R-d)^2}$$

When body is at depth d then only M' will apply gravitational force

$$M' = \frac{4}{3}\pi(R-d)^3 \rho$$

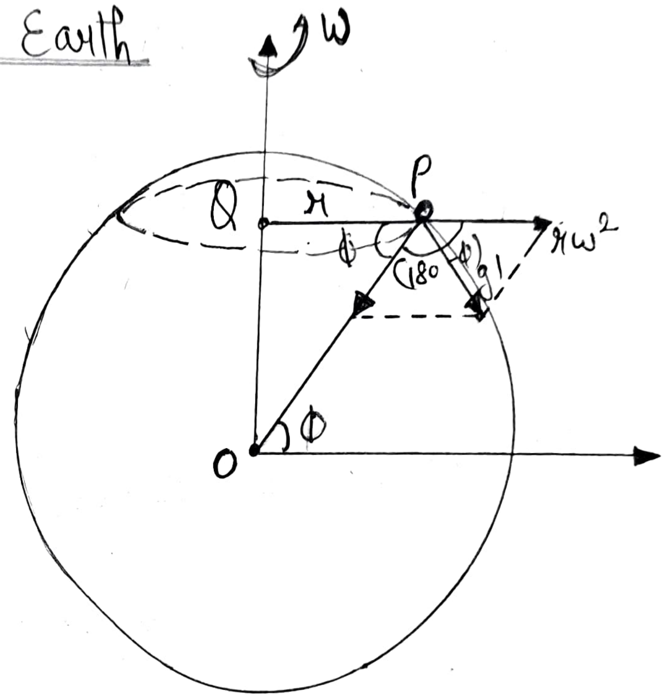
$$g' = \frac{\frac{4}{3} G\pi (R-d)^3 \rho}{(R-d)^2} \Rightarrow \boxed{g' = \frac{4}{3}\pi G\rho(R-d)} \quad (2)$$

Divide equation (2) by (1)

$$g' = \frac{g(R-d)}{R}$$

$$g' = g\left(1 - \frac{d}{R}\right)$$

IV Axial Rotation Of Earth

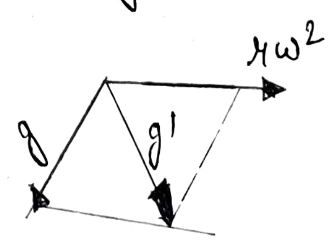


Let us consider an object of mass m is placed on the surface of earth at an angle of latitude ϕ

There are forces acting on the particle due to which the body is having acceleration in two direction

I. Acceleration due to gravity force which is g and acting towards earth centre

II. Since earth is revolving on its axis due to which the body is ~~ar~~ experiencing centrifugal force due to which body is accelerating with acceleration $r\omega^2$



Net acceleration g'

$$g' = \sqrt{(g)^2 + (r\omega^2)^2 + 2 g r\omega^2 \cos(180-\phi)}$$

Since $\omega = \frac{2\pi}{T} = \frac{2\pi}{24 \times 60 \times 60}$ = which is very small so ignoring $(r\omega^2)^2$

we get

$$g' = \sqrt{g^2 - 2 g r\omega^2 \cos\phi} \quad \text{—(1)}$$

Now in $\triangle OPQ$

$$\frac{r}{R} = \cos\phi$$

$$r = R\cos\phi \quad -(2)$$

Put value of (2) in (1)

$$g' = (g^2 - 2gR\omega^2\cos^2\phi)^{1/2}$$

$$g' = g\left(1 - \frac{2gR\omega^2\cos^2\phi}{g}\right)^{1/2}$$

$$g' = g\left[1 - \frac{1}{2}\left(\frac{2R\omega^2\cos^2\phi}{g}\right)\right]$$

$$g' = g\left[1 - \frac{R\omega^2\cos^2\phi}{g}\right]$$

$$g' = g - R\omega^2\cos^2\phi$$

at equator $\phi = 0^\circ$

so g' will be minimum

at pole $\phi = 90^\circ$

$\cos 90^\circ = 0$

so g' will be max at pole

GRAVITATIONAL FIELD

The space around a mass or system of masses in which any other test mass experiences a gravitational force is called gravitational field.

Test Mass ⇒ It is a particle which does not create its own gravitational field and its mass is nearly zero. This is a hypothetical particle.

Gravitational Field Strength | Intensity of Gravitational Field

The force experienced by a unit test mass placed at a point in a gravitational field is called gravitational field strength or intensity of gravitational field at that point.

$$\vec{E} = \lim_{m_0 \to 0} \frac{\vec{F}}{m_0}$$

where m_0 = Test mass

F = force experienced by test mass

E = Gravitational field strength

* It is a vector quantity

* Gravitational field strength is equal to acceleration produced in the test mass at that point

$$\vec{E} = \frac{\vec{F}}{m} \quad \text{—(1)}$$

$$\vec{a} = \vec{g} = \frac{\vec{F}}{m} \quad \text{—(2)}$$

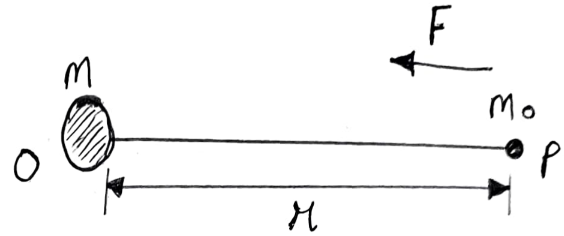
from (1) and (2)

$$\vec{g} = \vec{E}$$ OR = $g = E$

Gravitational Field due to a point Mass.

Let us consider a mass m which is placed at 'O' due to which we have to find gravitational field at point 'P' at a distance r from O.

Now in order to find out gravitational field we will place a test mass m_o at point P



Now force on test mass

$$F = \frac{GMm_o}{r^2}$$

Now gravitational field strength at point P

$$E = \frac{F}{m_o} = \frac{GMm_o}{r^2 m_o}$$

$$E = \frac{Gm}{r^2}$$

Direction of gravitational field is same as direction of force at that point

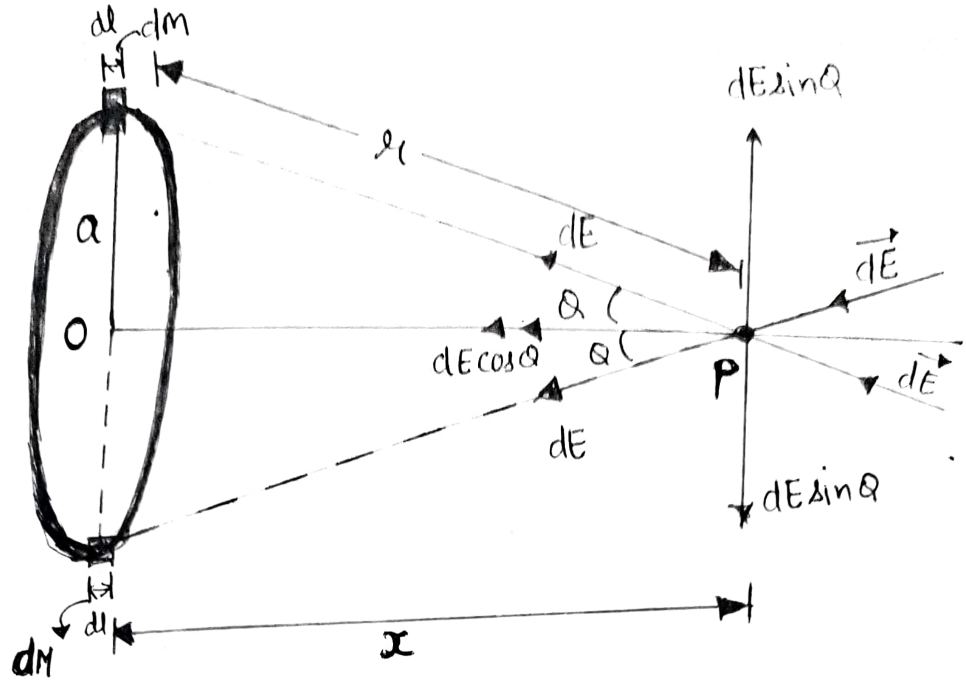
GRAVITATION FIELD DUE TO RING AT SOME POINT ON ITS AXES

Let us consider a ring of uniformly distributed mass 'm'. Let the radius of ring be 'a'. Let a Point 'p' at a distance x from the centre of ring at which we have to find out the value of gravitational field strength.

Consider a small element of ring dl of mass dm.

Now due to this element field at point P will be

$$dE = \frac{G\,dm}{r^2}$$



Now this $\vec{dE}$ will have two component

Along X axis = $dE\cos Q$

Along Y axis = $dE\sin Q$

This Y component of gravitational electric field will be cancelled out by Y component of gravitational field produced by a similiar diametrically opposite element of ring.

So total gravitational field at Point P

$$E_{total} = \int^{2\pi a} dE\cos Q \quad \text{———} \quad *$$

$$dE = \frac{Gdm}{r^2}$$

$$\cos Q = \frac{x}{r} \quad -(2)$$

$$dm = \frac{m}{2\pi a}dl$$

$$dE = \frac{Gm}{2\pi a r^2}dl \quad -(1)$$

Put value of ① and ② in ✱ equation

$$E_{total} = \int_0^{2\pi a} \left(\frac{Gm\,dl}{2\pi a r^2}\right)\frac{x}{r}$$

$$E_{total} = \frac{Gmx}{2\pi a r^3}\int_0^{2\pi a} dl$$

$$E_{total} = \frac{Gmx}{2\pi a r^3}\left[2\pi a - 0\right]$$

$$E_{total} = \frac{Gmx}{r^3}$$

$$r = \sqrt{a^2+x^2}$$

$$E_{total} = \frac{Gmx}{(a^2+x^2)^{3/2}}$$

Now gravitational field intensity at centre of Ring

$$x = 0$$

$$E = 0$$ Gravitational field strength will be zero at the centre of ring

Maximum value of Gravitional field Intensity

$$E = \frac{Gmx}{(a^2+x^2)^{3/2}}$$

For Maximum value $\frac{dE}{dx} = 0$

$$\frac{dE}{dx} = \frac{Gm(a^2+x^2)^{3/2} - Gmx\frac{3}{2}(a^2+x^2)^{\frac{1}{2}}2x}{(a^2+x^2)^3} = 0$$

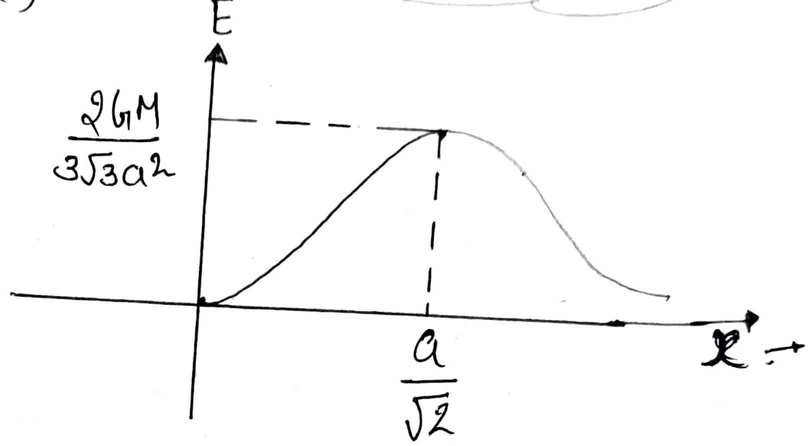
after solving above $x = \pm\frac{a}{\sqrt{2}}$

We will get Maximum at $x = \frac{a}{\sqrt{2}}$

So for maximum value of electric field intensity

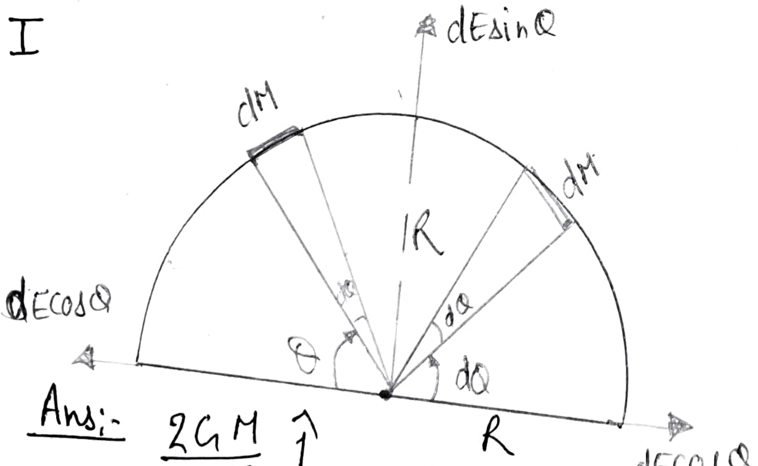
Put $x = \frac{a}{\sqrt{2}}$ in $E = \frac{Gmx}{(a^2+x^2)^{3/2}}$

So $E_{max} = \frac{Gm\left(\frac{a}{\sqrt{2}}\right)}{\left(a^2+\frac{a^2}{2}\right)^{3/2}} \Rightarrow \boxed{E_{max} = \frac{2GM}{3\sqrt{3}a^2}}$



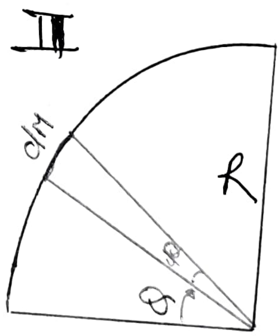
Q→ Find the value of electric field at the centre in following cases.
Mass in each case is M

I



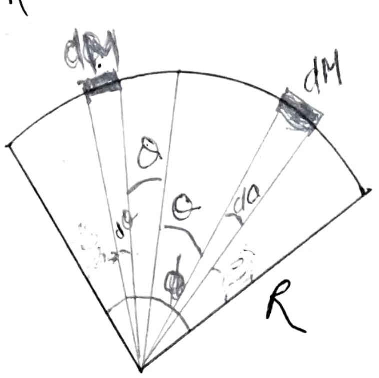
Ans:- $\frac{2GM}{\pi R^2}\,\hat{j}$

II

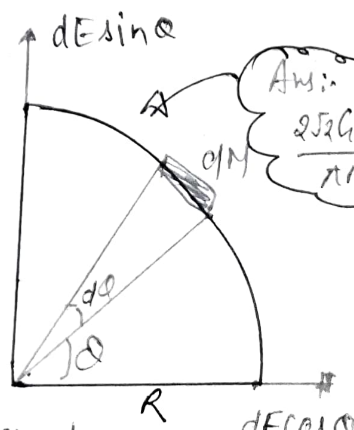


Ans→ $\frac{2\sqrt{2}GM}{\pi R^2}$

III



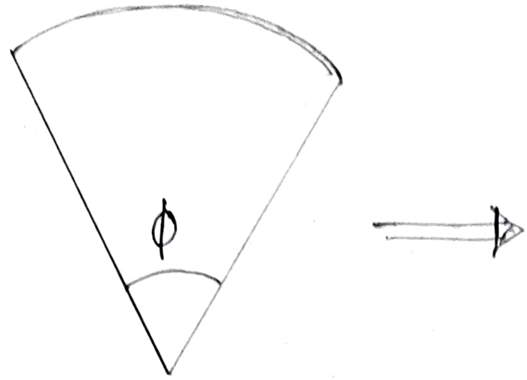
IV

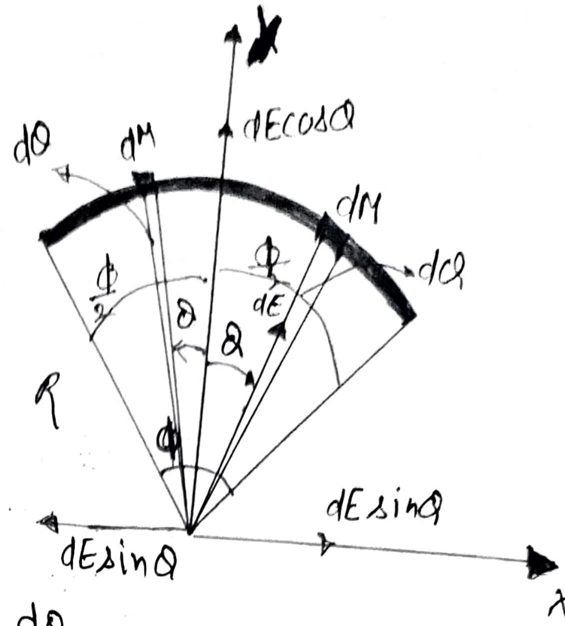


Ans:- $\frac{2\sqrt{2}GM}{\pi R^2}$

Hint→ You have to solve for x and y axis separately.

Sol (III)





Let us consider a small element of mass dM at an θ of angular width dθ
Now due to this element it will have two component of gravitational field

Along X axis ⇒ $dE\sin\theta$

along Y axis ⇒ $dE\cos\theta$

Now $dE\sin\theta$ component will be cancelled out by symmetrically opposite point so

Now total E net = $\int dE\cos\theta$

$R\phi \longrightarrow M$

$1 \longrightarrow \frac{M}{R\phi}$

$R\,d\theta \rightarrow \frac{M}{R\phi} R\,d\theta$

$\frac{M\,d\theta}{\phi}$

$$dE = \frac{G\,dM}{R^2}$$

$$dM = \frac{M\,d\theta}{\phi}$$

$$dE = \frac{GM\,d\theta}{R^2\phi}$$

$$E_{net} = \int_{-\frac{\phi}{2}}^{\frac{\phi}{2}} \frac{GM\,d\theta\cos\theta}{R^2\phi}$$

$$E_{net} = \frac{GM}{R^2\phi}\int_{-\frac{\phi}{2}}^{\phi/2}\cos\theta\,d\theta = \frac{GM}{R^2\phi}\left[\sin\theta\right]_{-\frac{\phi}{2}}^{\frac{\phi}{2}}$$

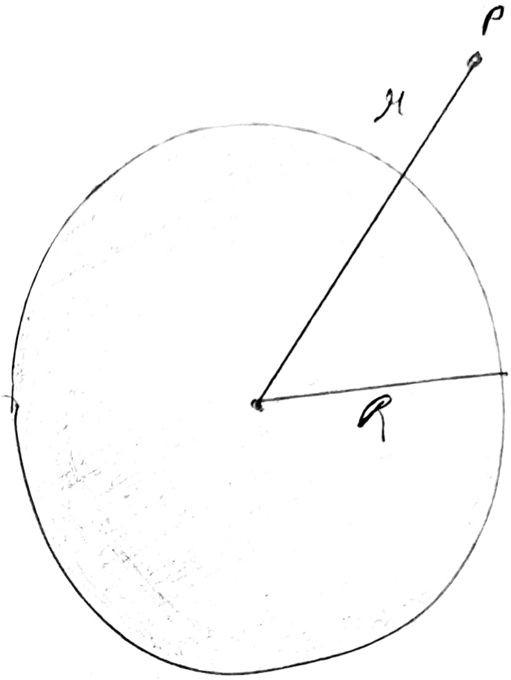
$$= \frac{GM}{R^2\phi}\left[\sin\frac{\phi}{2} - \sin\left(-\frac{\phi}{2}\right)\right]$$

$$\boxed{E_{net} = \frac{GM}{R^2\phi}\left[2\sin\frac{\phi}{2}\right]}$$

This general result for Ring

GRAVITATIONAL FIELD DUE TO SOLID SPHERE

(I) FIELD At any External Point P ($r > R$)



For any point out side sphere the sphere Can be treated as point size so

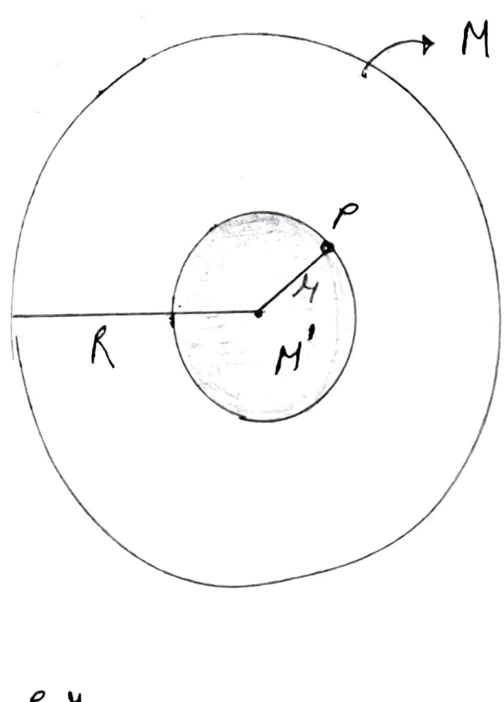
$$E = \frac{GM}{r^2} \qquad r > R$$

$$E \propto \frac{1}{r^2}$$ out side shell

II Field at any point on the sphere

$$E = \frac{GM}{R^2} \qquad r = R$$

(III) Field at any point Point inside the sphere



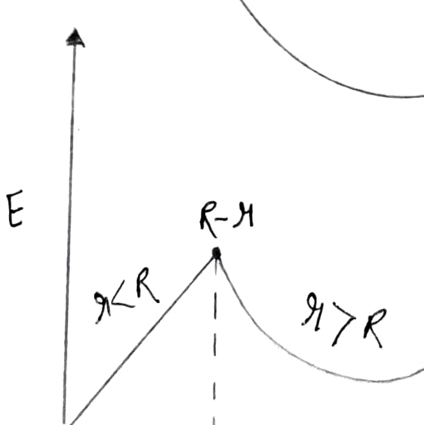
$$E = \frac{GM'}{r^2}$$

$$M' = \frac{M}{\frac{4}{3}\pi R^3} \times \frac{4}{3}\pi r^3$$

$$M' = \frac{Mr^3}{R^3}$$

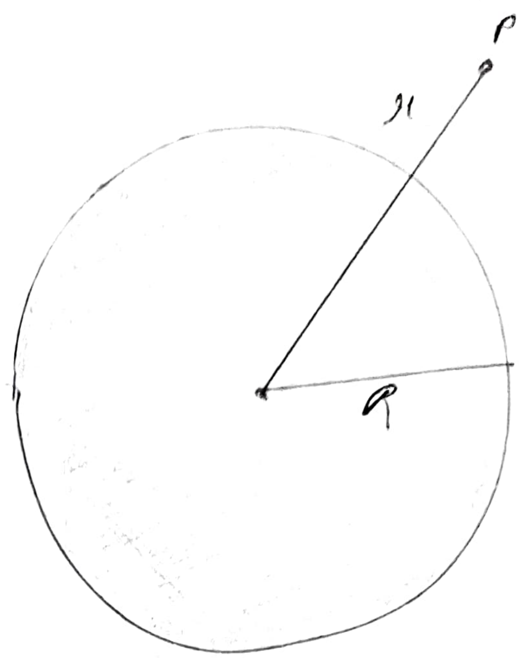
$$E = \frac{GMr^3}{r^2 R^3}$$

$$E = \frac{GMr}{R^3}$$



Gravitational Field Due To Solid Sphere

(I) Field At any External Point P ($r > R$)



For any point out side sphere the sphere Can be treated as point size ϕ 0

$$E = \frac{GM}{r^2} \qquad r > R$$

$$\boxed{E \propto \frac{1}{r^2}}$$ out side shell

II Field at any point on the sphere

$$E = \frac{GM}{R^2} \qquad r = R$$

(III) Field at any point Point inside the sphere

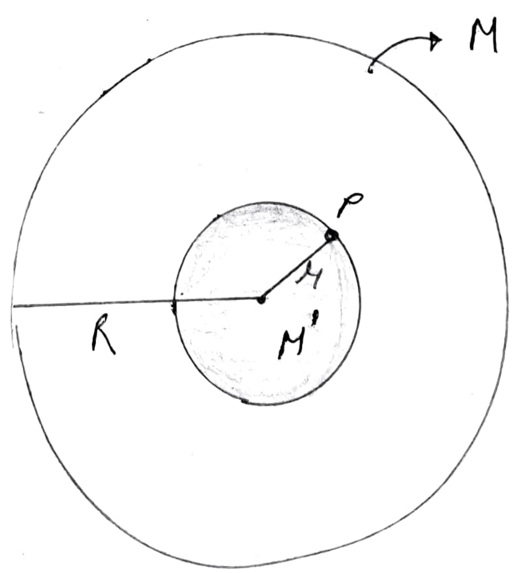
$$E = \frac{GM'}{r^2}$$

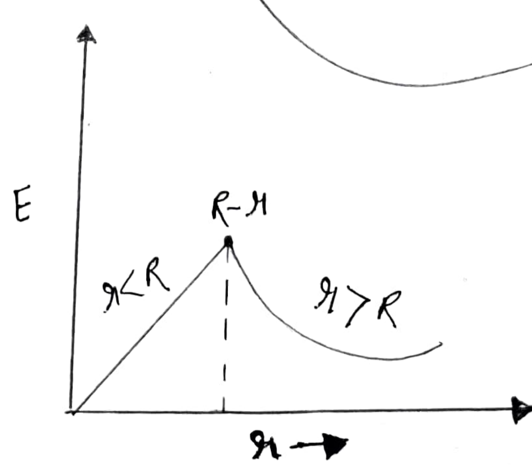
$$M' = \frac{M}{\frac{4}{3}\pi R^3} \times \frac{4}{3}\pi r^3$$

$$M' = \frac{Mr^3}{R^3}$$

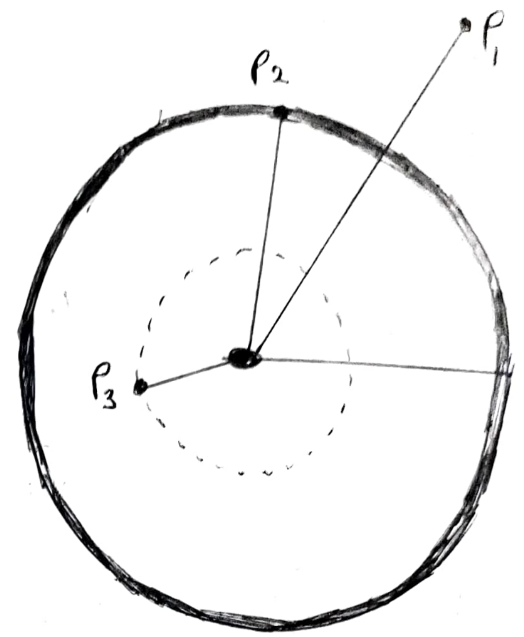
$$E = \frac{GMr^3}{r^2 R^3}$$

$$\boxed{E = \frac{GMr}{R^3}}$$





GRAVITATIONAL FIELD STRENGTH DUE TO SHELL



I. Out side Shell [$r > R$] [P_1]

In this case the whole shell can be treated as point mass so

$$\boxed{E = \frac{GM}{r^2}} \quad [r > R]$$

$$E \propto \frac{1}{r^2}$$

(II) At the surface of shell [$r = R$] [P_2]

$$\boxed{E = \frac{GM}{R^2}}$$

III At any point inside shell [P_3]
$r < R$

$$\boxed{E = 0}$$